DF/IST-7.2002

Noncommutative Scalar Field Coupled to Gravity

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(Dated: 7th February 2008)

A model for a noncommutative scalar field coupled to gravity is proposed via an extension of the Moyal product. It is shown that there are solutions compatible with homogeneity and isotropy to first non-trivial order in the perturbation of the star-product, with the gravity sector described by a flat Robertson-Walker metric. We show that in the slow-roll regime of a typical chaotic inflationary scenario, noncommutativity has negligible impact.

PACS numbers: 11.10.Lm, 04.50+h, 98.80-Cq

I. INTRODUCTION

The idea of noncommuting spatial coordinates has been proposed long ago by Snyder [1], when Quantum Field Theory itself had just been successfully established. More recently, it has been pointed out by Seiberg and Witten [2], that noncommutative geometry arises naturally in certain limits of string theory, which has motivated great interest in this topic. Given the Moyal deformation of the product of functions, which defines a noncommutative algebra, field theories have subsequently been constructed (see e.g. Refs. [3, 4] for reviews) even though issues like unitarity [5] and renormalizability are not yet fully established. On the other hand, realistic particle physics models have been considered, allowing to constrain, to some extent, the value of the noncommutative parameter [6].

These results rely mostly on a noncommutative algebra provided by a new product of functions. Noncommutative geometry has been systematized most remarkably by Connes [7] and Woronowicz [8], via the generalized concept of differential structure of generic ($C^\star$)-algebras. This formulation is believed to be the way to construct a noncommutative version of gravity and even quantum gravity, via noncommutative differential calculus [9]. In this work however, we shall not follow this approach but rather derive our results from the noncommutative algebra of a generalized multiplication law for tensors while leaving the Hilbert-Einstein action unchanged. It is believed that this noncommutative algebra approach may provide some insight into the quantum gravity physics at the Planck scale.

Several authors have studied the impact of new physics in inflationary models and, hence, on the gaussian character of energy density fluctuations or on the isotropy of the observables. Noncommutativity of the coordinates introduces a fundamentally new length scale whose imprint may be important [10, 11]. As will be discussed, our approach is similar to the study carried out in Ref. [12], even though differences in details lead to somewhat different conclusions. Most remarkably we find that our perturbation approach allows for an homogeneous and isotropic description of the impact of noncommutativity.

II. GENERALIZED MOYAL PRODUCT

Noncommutativity is usually introduced in Minkowski space via a noncommutative Moyal product defined as

$$T * W\,(x) = \sum_{n=0}^{\infty} \frac{(i/2)^n}{n!}\theta^{\alpha_1\beta_1}\dots\theta^{\alpha_n\beta_n}\left(T_{,\alpha_1\dots\alpha_n}\right)\left(W_{,\beta_1\dots\beta_n}\right), \tag{1}$$

where T and W are generic tensors whose indices have been suppressed, the primes denote partial derivatives and $\theta^{\alpha\beta}$ is constant. Aiming to preserve Lorentz symmetry we regard $\theta^{\alpha\beta}$ as an antisymmetric Lorentz tensor. This yields the commutator between coordinates

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}. \tag{2}$$

In order to preserve general covariance we introduce instead the following generalized Moyal product

$$T * W\,(x) = \sum_{n=0}^{\infty} \frac{(i/2)^n}{n!}\theta^{\alpha_1\beta_1}\dots\theta^{\alpha_n\beta_n}\left(T_{;\alpha_1\dots\alpha_n}\right)\left(W_{;\beta_1\dots\beta_n}\right), \tag{3}$$

where the semicolon denotes covariant derivative with the Levi-Civitta connection and $\theta^{\alpha\beta}$ is a non-constant rank-2 antisymmetric tensor. Despite being non-associative in general, we shall see that for a scalar field Φ such that $\theta^{\alpha\beta}\Phi_{;\alpha} = 0$ we recover associativity of the product to some extent.

By making use of the antisymmetry of $\theta^{\alpha\beta}$ one can easily prove that, under conjugation, $(T * W)^* = W^* * T^*$. The compatibility of the metric yields $g^{\mu\nu} * T = g^{\mu\nu}T$ so that the process of raising and lowering indices is not affected by noncommutativity.

Noncommutative Lagrangian densities are obtained by turning usual products into star-products so that one has to evaluate integrals of the form

$$S = \int d^4x\sqrt{-g}T^* * W. \tag{4}$$

Through the process of integration by parts and dropping surface terms we can arrange the covariant derivatives on the star-product to act either on T or on W,

that is

$$S = \int d^4x\sqrt{-g}T^*(\mathcal{A}W) \quad (5)$$
$$= \int d^4x\sqrt{-g}(\mathcal{A}T)^* W,$$

where $\mathcal{A}$ is an Hermitian operator given by

$$\mathcal{A}W = \quad (6)$$
$$\sum_{n=0}^{\infty} \frac{(-i/2)^n}{n!}\left[\theta^{\alpha_1\beta_1}\dots\theta^{\alpha_n\beta_n}\left(W_{;\beta_1\dots\beta_n}\right)\right]_{;\alpha_n\dots\alpha_1}.$$

If the Lagrangian density is quadratic on the tensor T one can use the property under conjugation to prove that $T*T$ is real; hence

$$S' = \int d^4x\sqrt{-g}T*T = \int d^4x\sqrt{-g}T\left(\frac{\mathcal{A}+\mathcal{A}^*}{2}\right)T$$
$$= \int d^4x\sqrt{-g}T\mathcal{O}T, \quad (7)$$

where the Hermitian operator $\mathcal{O} = \frac{1}{2}(\mathcal{A}+\mathcal{A}^*)$ has been introduced

$$\mathcal{O}W = \quad (8)$$
$$\sum_{n=0}^{\infty} \frac{(-1/4)^n}{(2n)!}\left[\theta^{\alpha_1\beta_1}\dots\theta^{\alpha_{2n}\beta_{2n}}\left(W_{;\beta_1\dots\beta_{2n}}\right)\right]_{;\alpha_{2n}\dots\alpha_1}.$$

III. NONCOMMUTATIVE SCALAR FIELD COUPLED TO GRAVITY

A. Massive scalar field

The noncommutative action for a massive scalar field, Φ, is quadratic, and so, according with the results from the last section

$$S = -\tfrac{1}{2}\int d^4x\sqrt{-g}\left\{\nabla^\mu\Phi\mathcal{O}\nabla_\mu\Phi - m^2\Phi\mathcal{O}\Phi\right\}, \quad (9)$$

and the equation of motion being given by

$$\nabla^\mu\mathcal{O}\nabla_\mu\Phi + m^2\mathcal{O}\Phi = 0. \quad (10)$$

Operator $\mathcal{O}$ naturally arises in the equation of motion. Furthermore, being an Hermitian operator it must correspond to an observable of the scalar field Φ. In the commutative limit, $lim_{\theta\to 0}\mathcal{O} = 1$. On the other hand, switching off gravity and admitting that $\theta^{\alpha\beta}$ is constant yields $\mathcal{O} = 1$, since the partial derivatives commute and are contracted with the antisymmetric tensor $\theta^{\alpha\beta}$ in (8). Hence, in this model, noncommutativity arises only for non-trivial gravity. This is due to the fact that the usual Moyal product obeys, under integration, the cyclic property

$$\int d^4x\, f*g = \int d^4x\, f\, g = \int d^4x\, g*f \quad (11)$$

and the action we are considering is quadratic.

B. Scalar field with an arbitrary potential

In this section we consider the noncommutative generalization of an arbitrary analytic commutative potential $V(\Phi)$. Associativity played no part in the previous section because one dealt with a quadratic action. Now, however, the case of an arbitrary potential requires special care as, in general, the resulting star-product is not associative.

Given a commutative analytic potential

$$V(\Phi) = \sum_{n=0}^{\infty}\frac{\lambda_n}{n!}\Phi^n \quad (12)$$

one wishes to consider a noncommutative potential $V_{NC}(\Phi)$ in the form

$$V_{NC}(\Phi) = \sum_{n=0}^{\infty}\frac{\lambda_n}{n!}\overbrace{\Phi*\dots*\Phi}^{n\ factors}. \quad (13)$$

provided that the action of the star-product upon powers of the scalar field Φ is associative. We shall analyze this generalization of the potential in the case where $\theta^{\alpha\beta}\Phi_{;\beta} = 0$.

Since there is no a priori associativity, let us consider the sequence

$$s_2 = (\Phi*\Phi) \qquad s_{n+1} = \Phi*s_n, \quad n>2. \quad (14)$$

It is easy to prove that, up to second order

$$s_n \simeq \Phi^n + \frac{n(n-1)}{2}\Phi^{n-2}(\Phi\hat{*}\Phi) \quad (15)$$

where one defines

$$\varphi\hat{*}\chi = -\tfrac{1}{8}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\left(\varphi_{;\alpha_1\alpha_2}\right)\left(\chi_{;\beta_1\beta_2}\right). \quad (16)$$

Also, for every m and n, it can be shown that, up to second order

$$s_n * s_m \simeq s_{m+n} \quad (17)$$

which proves that one can calculate the power s_q grouping q star-products in any way one wishes. Hence the star-product of powers of Φ is associative under these conditions. Also, these results enables one to write

$$V_{NC}(\Phi) \equiv V(\Phi) + \tfrac{1}{2}V''(\Phi)(\Phi\hat{*}\Phi), \quad (18)$$

where $' = d/d\Phi$.

The variation of the potential V_{NC} in the action is given by

$$-\frac{\delta S_{pot}}{\delta\Phi} = V'(\Phi) + \tfrac{1}{2}V'''(\Phi)(\Phi\hat{*}\Phi) - \tfrac{1}{4}\mathcal{F}[V,\Phi], \quad (19)$$

where we have defined the operator

$$\mathcal{F}[V,\Phi] = \left[\tfrac{1}{2}V''\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\phi_{;\beta_1\beta_2}\right]_{;\alpha_2\alpha_1}. \quad (20)$$

With this definition we also obtain that

$$\mathcal{O}\Phi_{;\mu} \simeq \Phi_{;\mu} - \tfrac{1}{8}\mathcal{F}\left[\Phi^2, \Phi_{;\mu}\right]. \quad (21)$$

C. Homogeneous and isotropic space-time

In what follows we shall assume that the Einstein-Hilbert action is unchanged by noncommutativity. This is done as our aim is the study of the implications of a noncommutative algebra of tensors and because there is no canonical way of introducing this algebra in the geometrical formulation of gravity (i.e. in the Riemann tensor and, ultimately, in the Ricci scalar). Hence the Einstein equations are given by

$$R_{\alpha\beta} = -8\pi k\left[\tfrac{1}{2}\nabla_{\{\alpha}\Phi\mathcal{O}\nabla_{\beta\}}\Phi + g_{\alpha\beta}V_{NC}(\Phi)\right]. \quad (22)$$

As a concrete model we consider a homogeneous and isotropic space-time described by the spatially flat Robertson-Walker metric

$$ds^2 = -dt^2 + R^2(t)\left(dx^2 + dy^2 + dz^2\right). \quad (23)$$

The non-vanishing components of the Christoffel symbols are given by

$$\Gamma^t{}_{ij} = R\dot{R}\delta_{ij} \qquad \Gamma^i{}_{jt} = \frac{\dot{R}}{R}\delta^i{}_j \quad (24)$$

and the Ricci tensor is diagonal and given by

$$R_{tt} = 3\frac{\ddot{R}}{R} \qquad R_{ij} = -\left(R\ddot{R} + 2\dot{R}^2\right)\delta_{ij}. \quad (25)$$

The non-vanishing components of the antisymmetric noncommutative tensor, $\theta^{\alpha\beta}$, correspond to two 3-vectors which we denote by $\vec{E}$ and $\vec{B}$, in analogy with the electromagnetic tensor. Thus, even if $\theta^{\alpha\beta}$ is homogeneous, $\theta^{\alpha\beta} = \theta^{\alpha\beta}(t)$, it is still possible that symmetry under rotations is broken and some care should be taken concerning the choice of an isotropic *Ansatz* of the metric, since $\vec{E}$ and $\vec{B}$ can induce preferred directions in space. We shall show, however, that there is a noncommutative model consistent with homogeneity and isotropy to first order in perturbation theory, for the homogeneous scalar field $\partial_i\Phi = 0$. Under these conditions, we get from Eqs. (10) and (22)

$$\ddot{\Phi} + 3\frac{\dot{R}}{R}\dot{\Phi} + V' = \quad (26)$$
$$\frac{\partial_t\left(R^3\mathcal{F}\left[\Phi^2,\Phi_{;t}\right]\right)}{8R^3} + \tfrac{1}{2}V'''(\Phi\hat{*}\Phi) + \tfrac{1}{4}\mathcal{F}[V,\Phi],$$

$$\left(\frac{\dot{R}}{R}\right)^2 = \quad (27)$$
$$\frac{8\pi k}{3}\left(\tfrac{1}{2}\dot{\Phi}^2 + V + \tfrac{1}{2}V''(\Phi\hat{*}\Phi) - \tfrac{1}{16}\dot{\Phi}\mathcal{F}\left[\Phi^2,\dot{\Phi}\right]\right).$$

We show in the Appendix the explicit computation of one of these terms, the remaining ones being analogous

$$\begin{aligned}\Phi\hat{*}\Phi &= -\tfrac{1}{2}\left(R\dot{R}\dot{\Phi}B\right)^2,\\ \mathcal{F}[V,\Phi] &= \tfrac{1}{2R^3}\partial_t\left[R^5\dot{R}^2\dot{\Phi}B^2\tfrac{1}{2}V''\right],\\ \mathcal{F}\left[\Phi^2,\dot{\Phi}\right] &= -\tfrac{2}{R^3}\partial_t\left[R^6\dot{R}^2B^2\partial_t\left(\tfrac{\dot{\Phi}}{R}\right)\right],\end{aligned} \quad (28)$$

where we used the condition $\vec{E} = 0$ [14]. This condition ensures that we have $\theta^{\alpha\beta}\Phi_{;\beta} = 0$ and the noncommutative generalization of the scalar potential (18) makes sense. Hence we see that the dependence of Eqs. (28) in $\theta^{\alpha\beta}$ is only through B^2 and consequently invariance under rotations is preserved. Since there is no known dynamics for the $\vec{B}$ field, we consider the relationship

$$B^2 = \hat{B}^2R^{-2\varepsilon}, \quad (29)$$

where $\hat{B}^2$ is a constant. We shall determine the parameter ε in the next section.

IV. SLOW-ROLL IN CHAOTIC INFLATION

Here we investigate the implications of noncommutativity in the slow-roll phase of a typical chaotic inflation. Given its generic features and the fairly general conditions for the onset of inflation, chaotic models [13] are particularly suited for studying the effect of noncommutativity. We seek solutions of Eqs. (26) and (27) in first order of perturbation theory in $\hat{B}^2$. To perform this we consider solutions of the following form

$$\Phi = \phi + \hat{B}^2\varphi \qquad R = a + \hat{B}^2\chi, \quad (30)$$

where ϕ and a are solutions of the unperturbed (commutative) problem, while φ and χ are arbitrary time dependent functions to be determined. We ignore in every step higher order terms in $\hat{B}^2$. Using units in which $k = 1$, Eqs. (26) and (27) take compact form

$$\ddot{\Phi} + 3\frac{\dot{R}}{R}\dot{\Phi} + V' = \hat{B}^2f, \quad (31)$$

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}\left(\tfrac{1}{2}\dot{\Phi}^2 + V\right) + \frac{8\pi}{3}\hat{B}^2g \quad (32)$$

in terms of functions f and g specified below. Standard perturbation theory gives rise to the usual inflationary equations

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0, \tag{33}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{34}$$

Onset of inflation and slow-roll regime are achieved once the following conditions are met

$$\frac{V'}{V} \leq \sqrt{48\pi} \quad , \quad \frac{V''}{V} \leq 24\pi, \tag{35}$$

so that we can neglect $\ddot{\phi}$ in the Eq. (33) and $\dot{\phi}^2/2$ in Eq. (34). It then follows the useful condition

$$\left|\dot{\phi}\right| \leq \sqrt{2}V^{1/2}. \tag{36}$$

Therefore, terms in Eqs. (28) can be evaluated using the slow-roll conditions and, as illustrated in the Appendix, we obtain that all of them are proportional to $a^{4-2\varepsilon}$ and to factors that depend on V and $\dot{\phi}$. Since the Universe is expanding at an exponential rate, the perturbation theory is meaningful only if $\varepsilon \geq 2$. However, $\varepsilon > 2$ implies that the terms in Eqs. (28) decrease so rapidly that noncommutativity will not lead to any effect. Thus, we conclude from the consistency of perturbation theory that $\varepsilon = 2$. This is a natural choice from the theoretical point of view as well. Most of the studied noncommutative models use a constant $\theta^{\alpha\beta}$. If one requires that this is so for the physical coordinates $y^i = R\,x^i$, then one finds, inspired in Eq. (2), $\left[y^i, y^j\right] = \hat{B}^{ij}$ and hence $\varepsilon = 2$.

Equations for the perturbations are obtained gathering all terms proportional to $\hat{B}^2$ and this constant cancels out from the differential equations. Function φ satisfies the relationship

$$f - 4\pi\frac{\dot{\phi}}{\dot{a}/a}g = \tag{37}$$
$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\left[1 + \frac{4\pi}{3}\left(\frac{\dot{\phi}}{\dot{a}/a}\right)^2\right]\dot{\varphi} + V''\left[1 + 4\pi\frac{V'}{V''}\frac{\dot{\phi}}{\dot{a}/a}\right]\varphi,$$

where functions f and g can be estimated (see Appendix) using the slow-roll conditions:

$$|f| \leq a_1\tfrac{1}{2}V^2V'' + a_2\tfrac{1}{2}V^2V''' + a_3V^3 + a_4V^{5/2},$$
$$|g| \leq a_5V^3 + a_6\tfrac{1}{2}V^2V'', \tag{38}$$

with $a_1 \simeq 85.5$, $a_2 \simeq a_6 \simeq 4.2$, $a_3 \simeq 3.30 \times 10^3$, $a_4 \simeq 4.52 \times 10^3$ and $a_5 \simeq 1.76 \times 10^2$.

Potentials in chaotic inflation are characterized by a small overall coupling constant, $\lambda \simeq 10^{-14}$, to ensure consistency with the amplitude of energy density perturbations around 10^{-5}, for ϕ field values of a few Planck units. Writing the potential as

$$V(\Phi) = \lambda\, v(\Phi), \tag{39}$$

and considering

$$v \leq 10^2, \tag{40}$$

it then follows that $|f| \leq 4.5{\times}10^{-27}$ and $|g| \leq 1.8{\times}10^{-34}$. However, the second term of the right-hand side of Eq. (37) is of the order $2.5{\times}10^{-6}$ while the third term is of the order $7{\times}10^{-11}$. So, for numerical purposes, the left-hand side of the Eq. (37) is vanishingly small. But, in this case, one gets essentially the same differential equation that would arise when performing perturbation theory on the standard slow-roll approximation with no extra physics. Therefore, we conclude that noncommutativity introduces no change in inflationary slow-roll physics for the inflaton field.

Furthermore, from the equation for the χ perturbation

$$\frac{d}{dt}\left(\frac{\chi}{a}\right) = \frac{4\pi}{3\dot{a}/a}\left(\dot{\phi}\dot{\varphi} + V'\varphi + g\right) \tag{41}$$

we can see that the upper limit for $|g|$ assures that, numerically, there is no difference in this equation as well. Hence, we are led to conclude that performing perturbation calculations on the presence of noncommutativity has no effect in the standard chaotic inflationary model.

V. CONCLUSIONS

In this work we have considered a natural extension of the Moyal product and studied its implications to the physics of a scalar field coupled to gravity. We presented the general features of our formalism and applied it to a specific study of the scalar field on a spatially flat Robertson-Walker metric.

Our results were obtained using perturbation theory, which necessarily requires that the antisymmetric noncommutative tensor, $\theta^{\alpha\beta}$, is small compared to the covariant derivative of the fields. This can be seen analyzing the explicit expression for the Moyal product, Eq. (3). Despite the fact that there is no equation for $\theta^{\alpha\beta}$, both perturbation theory and theoretical considerations show that $\theta^{\alpha\beta} \sim R^{-2}$, where R is the scale factor. So we find that once perturbation theory is valid, it remains so as far as the expansion of the Universe lasts.

The antisymmetric tensor $\theta^{\alpha\beta}$ can be parameterized by two three-vectors, just like in the case of the electromagnetic tensor (c.f. Eq. (42) below). The homogeneity requirement, that is, $\partial_i\theta^{\alpha\beta} = 0$, could still lead to preferred directions in space rendering the isotropic Ansatz of the Robertson-Walker metric meaningless. We show, in first order perturbation theory, this is not the case

since the terms arising from noncommutative contributions depend only on the rotationally invariants E^2 and B^2.

In the slow-roll regime in the context of a typical chaotic inflation we show that noncommutativity introduces negligible corrections. This is mainly due to two reasons. First, the scale parameter (that might yield large deviations) does not appear in the first order terms as $\theta^{\alpha\beta} \sim R^{-2}$, otherwise these would grow exponentially and perturbation theory would be meaningless. On the other hand, the slow-roll conditions yield small derivatives for the inflaton field, Eq. (36), and for the logarithm of the scale factor, Eq. (34), given the small coupling constant of the inflaton potential. Since the Moyal product involves many derivatives (it is highly non-local) the smallness of the noncommutative contributions is well understood. We can look at this from another perspective: since perturbation theory requires that $\theta^{\alpha\beta}$ is small compared to the derivatives and these are very small, then noncommutativity is naturally suppressed. Therefore we are led to conclude that noncommutative effects, if any, necessarily arise in the non-perturbative regime of the theory.

Some remarks on our perturbative approach are in order. Our calculations assume that perturbation theory is valid from a given cosmological time t_* onward. Thus, if the conditions for inflation are met and assuming that $B = \hat{B}R^{-2}$, noncommutativity plays a negligible role. This implies that $B_* = \hat{B}R_*^{-2} \ll 1$, that is $\hat{B} \ll R_*^2$, and therefore, a sufficiently small $\hat{B}$ garantees the validity of perturbation theory for any given R_*. The actual value for $\hat{B}$ can be quite small if t_* is the onset of inflation, a time characterized by a small R_*. Notice that the constant $\hat{B}$ cancels out in the perturbative differential equations, so its smallness plays no role on the smallness of the extra terms in the perturbative differential equations (37) and (41). These terms are small because they involve derivatives of high degree and powers of the scalar potential, which has a small coupling constant.

As for the behavior of the noncommutative tensor prior to t_*, we propose no model for B. Even if the expression $B = \hat{B}R^{-2}$ or other one with a singularity for B at $R = 0$ apply, this clearly occurs before perturbation theory is valid. Furthermore, if t_* coincides with the onset of inflation then the physics prior to t_* has small impact, since chaotic initial conditions are assumed.

There is also another scenario where our conclusions remain valid. If the nonperturbative regime allows for inflation then it could be that part of inflation is initialy driven by noncommuativity and, at a later time, driven by the mechanism here described. This would allow a perturbative treatment beginning at a later time t_* so that the scale factor R_* would be larger by several orders of magnitude, and would allow larger values for $\hat{B}$, also by several orders of magnitude.

VI. APPENDIX

Here we perform some of the explicit calculations leading to the invariance of the theory under rotations, and illustrate the small corrections that noncommutativity introduces in the slow-roll regime. We recall that $\partial_i\Phi = 0$ and $\partial_i\theta^{\alpha\beta} = 0$ so that all spatial derivatives vanish. We use the notation

$$\theta^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{42}$$

Our aim is to calculate, for instance

$$\Omega \equiv \left[\tfrac{1}{2}V''\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\Phi_{;\beta_1\beta_2}\right]_{;\alpha_2;\alpha_1} = T^{\alpha_1\alpha_2}{}_{;\alpha_2\alpha_1}, \tag{43}$$

where the definition of the tensor $T^{\alpha_1\alpha_2}$ is evident.

It is easy to show that

$$T^{\alpha_1\alpha_2}{}_{;\alpha_2\alpha_1} = \frac{\partial_{\alpha_1}\left(\sqrt{-g}T^{\alpha_1\alpha_2}{}_{;\alpha_2}\right)}{\sqrt{-g}} = \frac{\partial_t\left(R^3T^{t\alpha_2}{}_{;\alpha_2}\right)}{R^3}, \tag{44}$$

and hence

$$\begin{aligned} T^{t\alpha_2}{}_{;\alpha_2} &= \frac{\partial_{\alpha_2}\left(\sqrt{-g}T^{t\alpha_2}\right)}{\sqrt{-g}} + \Gamma^t{}_{\alpha_2\nu}T^{\alpha_2\nu} \\ &= \frac{\partial_t\left(R^3T^{tt}\right)}{R^3} + R\dot{R}\delta_{ij}T^{ij}. \end{aligned} \tag{45}$$

To evaluate the relevant entries of the tensor $T^{\alpha_1\alpha_2}$ we compute

$$S_{\mu\nu} \equiv \Phi_{;\nu\mu} = \ddot{\Phi}\delta^t_\mu\delta^t_\nu - R\dot{R}\dot{\Phi}\tilde{\delta}_{\mu\nu} \tag{46}$$

to get

$$\begin{aligned} T^{tt} &= -\tfrac{1}{2}R\dot{R}\dot{\Phi}V''\delta_{\mu\nu}\theta^{t\mu}\theta^{t\nu} = -\tfrac{1}{2}R\dot{R}\dot{\Phi}V''E^2, \\ \delta_{ij}T^{ij} &= \tfrac{1}{2}V''\left[\delta_{ij}\theta^{it}\theta^{it}\ddot{\Phi} - R\dot{R}\dot{\Phi}\delta_{ij}\theta^{i\beta_1}\theta^{j\beta_2}\delta_{\beta_1\beta_2}\right] \\ &= \tfrac{1}{2}V''\left[E^2\ddot{\Phi} - 2R\dot{R}\dot{\Phi}B^2\right]. \end{aligned} \tag{47}$$

Thus we can explicitly verify the invariance of these expressions under rotations as well as under translations.

Taking $\vec{E} = 0$ leads to

$$\Omega = -\frac{1}{R^3}\partial_t\left[R^5\dot{R}^2\dot{\Phi}B^2V''\right]. \tag{48}$$

Using perturbation theory in the slow-roll approximation, Eqs. (33), (34) and (35), one is lead to evaluate

$$\begin{aligned} \Omega' &= -\frac{2}{a^3}\partial_t\left[a^{5-2\varepsilon}\dot{a}^2\dot{\phi}\tfrac{1}{2}V''\right] \\ &= -\frac{16\pi\dot{\phi}}{3a^3}\partial_t\left[a^{7-2\varepsilon}V\tfrac{1}{2}V''\right] \\ &= -\frac{16\pi a^{4-2\varepsilon}\dot{\phi}}{3}\left[(7-2\varepsilon)\sqrt{\tfrac{8\pi}{3}}V^{1/2}F + F'\dot{\phi}\right], \end{aligned} \tag{49}$$

where $F = \frac{1}{2}VV''$. Using Eq. (36) and $\varepsilon = 2$, it follows that

$$|\Omega'| \leq \frac{16\sqrt{2}\pi}{3}\,|V|\left[3\sqrt{\frac{8\pi}{3}}\,|F| + \sqrt{2}\,|F'|\right]. \qquad (50)$$

Finally, from Eqs. (39) and (40), one obtains

$$|\Omega'| \leq 2.4 \times 10^{-37}, \qquad (51)$$

which illustrates how small are the corrections introduced by noncommutativity.

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